

SLICES OF OKOUNKOV BODIES OF BIG DIVISORS ON MORI DREAM SPACES

JAESUN SHIN

ABSTRACT. The purpose of this paper is to study the slices of the Okounkov bodies of Mori dream spaces. First, we analyze all the slices of the Okounkov bodies of big divisors on Mori dream spaces associated to some admissible flags. As a byproduct, we obtain their descriptions on Mori dream threefolds. Finally, we consider its application to the rational polyhedrality of them.

1. INTRODUCTION

In this paper, we work over \mathbb{C} and assume that a variety is smooth unless otherwise stated. Let X be a projective variety, X_\bullet an admissible flag (see Definition 2.1) and D a big divisor on X . By the earlier works of Okounkov ([13], [14]), Lazarsfeld and Mustață ([11]) constructed the convex bodies of D on X associated to X_\bullet , denoted by $\Delta_{X_\bullet}(D)$, which we call the Okounkov bodies of D associated to X_\bullet . The importance of the Okounkov bodies of a big divisor is that we can study geometric problems through combinatoric ones. After the construction by Lazarsfeld and Mustață, there are many works to obtain the various properties of a divisor by using the Okounkov bodies. In particular, the Okounkov bodies are closely related to the numerical properties of a given big divisor. For example, it only depends on the numerical class of a big divisor (see [11, Proposition 4.1]). Also, Küronya and Lozovanu proved that nefness ([9, Corollary 2.2]) and ampleness ([9, Corollary 3.2]) of big divisors can be recovered from the shapes of the Okounkov bodies. Furthermore, they proved that the infinitesimal Okounkov bodies contain informations about moving Seshadri constant ([8, Theorem C]).

However, the explicit computations of the Okounkov bodies are difficult due to the complications unless X is low-dimensional or has simple structures. For example, when X is a curve, since a divisor D is just a point, we can easily obtain $\Delta_{X_\bullet}(D)$ by using Riemann-Roch (see Example 2.13). Also, when X is a surface, Zariski proved that every pseudo-effective divisor D has a Zariski decomposition ([11, Proposition 2.3.19]). By using it, we can describe the Okounkov bodies of big divisors associated to a flag $X_\bullet : X \supset C \supset \{x\}$ (Example 2.14) as the following shows (in fact, it can be extended to pseudo-effective divisors by using limiting Okounkov bodies (see [4, Theorem 4.5])),

$$\Delta_{X_\bullet}(D) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \text{ord}_C(\|D\|) \leq x_1 \leq \mu, \\ \text{ord}_x(N_{D-x_1C}|_C) \leq x_2 \leq \text{ord}_x(N_{D-x_1C}|_C) + (C.P_{D-x_1C})\},$$

where $\text{ord}_C(\|D\|)$ is the asymptotic valuation of D on C (Definition 2.16), $\mu := \sup\{s > 0 \mid D - sC \text{ is big}\}$, $D - x_1C = P_{D-x_1C} + N_{D-x_1C}$ is the Zariski decomposition, and $\text{ord}_x(N_{D-x_1C}|_C)$ is the order of $N_{D-x_1C}|_C$ at x .

Now, consider the case when X is a variety of dimension $n \geq 3$. The following question arises from the above description on a surface.

Question 1.1. Is there a general description of the Okounkov bodies of divisors on X as that on a surface?

One way to describe the Okounkov bodies is to analyze all the slices of them. In this way, we obtain the Okounkov bodies of a surface using the Zariski decomposition of a divisor. However, in higher dimensions, Zariski decompositions do not exist in general. Thus, when X is a variety of dimension $n \geq 3$, we can consider two cases. The first and the easiest case is when all the big divisors admit the Zariski decompositions. We can obtain their Okounkov bodies by following the same construction as that of a surface case. One of the simplest examples is a threefold whose pseudo-effective cone and nef cone coincide (see Example 2.19). In this case, by using the similar argument in the proof of Lemma 3.6, we can describe the Okounkov bodies. The next difficult case is when not all divisors on X have the Zariski decomposition. In general, it is hard to give an answer to Question 1.1 for the last case. Therefore, we consider it to the case when X is a Mori dream space (see Definition 2.21). One of the nice features of Mori dream spaces is that every divisor has a decomposition similar with a Zariski decomposition (see Proposition 2.24), which is helpful to obtain the Okounkov bodies.

The aim of this paper is to analyze all the slices of Okounkov bodies of big divisors on Mori dream spaces, which are essential to have an answer on Question 1.1. Now, let X be a Mori dream space of dimension n , D a big divisor on X , and $X_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}$ an admissible flag with $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, where $\text{Ud}(f_i)$ and I_D are defined in Note 3.4. The main idea is to observe $\Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D_t)$ and $\Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^*(f_{i*}^c D_t))$, where D_t , ϕ_i , $\tilde{X}_{i\bullet}$, and \tilde{f}_i are as in Notation 3.2 and Note 3.4. By Proposition 3.8, we obtain that they are the same. By using this with some lemmas, we obtain our main theorem.

Theorem 1.2. (=Theorem 3.10) *Let X , X_\bullet , D , I_D , and $[\alpha_i, \beta_i]$ be as in Notation 3.2 and Note 3.4. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), \dots, l_i^{n-1}(t))$ defined on each $[\alpha_i, \beta_i]$ such that*

$$\Delta_{X_\bullet}(D)_{x_1=t} = \Delta_{Y_1\bullet}(P_{D_t}|_{Y_1}) + l_i(t)$$

for all $t = t_i \in [\alpha_i, \beta_i]$.

Note that the linear function $l_i(t)$ on Theorem 1.2 is defined in the proof of Lemma 3.6. Theorem 1.2 says that if we know the Okounkov bodies of big divisors on Y_1 , we obtain all the slices of $\Delta_{X_\bullet}(D)$ on X so that $\Delta_{X_\bullet}(D)$ is known. Moreover, Theorem 1.2 can be extended to the limiting Okounkov bodies of pseudo-effective divisors naturally (Remark 3.12). Since we know the Okounkov bodies of big divisors on a surface, we obtain their descriptions on Mori dream threefolds as a byproduct of Theorem 1.2.

Corollary 1.3. (=Corollary 3.14) *Let X be a Mori dream threefold, and $X_\bullet : X \supset S \supset C \supset \{p\}$ an admissible flag with $S \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, where D , I_D , and $[\alpha_i, \beta_i]$ are as in Notation 3.2 and Note 3.4. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), l_i^2(t))$ defined on each $[\alpha_i, \beta_i]$ such that*

$$\Delta_{X_\bullet}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{ord}_S(\|D\|) \leq x_1 \leq \mu, \text{ for each } x_1 = t_i \in [\alpha_i, \beta_i], \\ l_i^1(t_i) \leq x_2 \leq \mu_{t_i} + l_i^1(t_i), \delta_{x_2}(t_i) \leq x_3 \leq \delta_{x_2}(t_i) + (P_{(P_{D_{t_i}}|_{S-x_2C})} \cdot C)\},$$

where $\mu = \sup\{t > 0 \mid D - tS \in \text{Big}(X)\}$, $\mu_t = \sup\{\alpha > 0 \mid P_{D_t}|_S - \alpha C \in \text{Big}(S)\}$, $P_{D_{t_i}}|_{S-x_2C} = P_{(P_{D_{t_i}}|_{S-x_2C})} + N_{(P_{D_{t_i}}|_{S-x_2C})}$ is the Zariski decomposition in the usual sense, and $\delta_{x_2}(t_i) = \text{ord}_p(N_{(P_{D_{t_i}}|_{S-x_2C})}|_C) + l_i^2(t_i)$ is a linear function on each $[\alpha_i, \beta_i]$ for fixed $x_2 \in [l_i^1(t_i), \mu_{t_i} + l_i^1(t_i)]$.

Moreover, Corollary 3.16 says that Corollary 1.3 also holds for a pseudo-effective divisor D . Then, the natural question one can ask is whether the description in Corollary 1.3 holds without the assumption that $S \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$. However, we can see that it does not hold in general (Caution 3.15).

The paper is organized as follows. In Section 2, we recall the construction of Okounkov bodies, that of the restricted Okounkov bodies, and some examples. Moreover, we recall the definition of a Mori dream space and their basic properties. Section 3 is the main part of this paper. In this section, we prove Theorem 1.2, and describe the Okounkov bodies of big divisors on Mori dream threefolds. Finally, in Section 4, we give an application of Corollary 1.3 to the rational polyhedrality of the (limiting) Okounkov bodies of Mori dream threefolds.

Notation 1.4.

- (1) $\text{Pic}(X)$: the group of Cartier divisors on X modulo linear equivalence.
- (2) $N^1(X)$: the group of Cartier divisors on X modulo numerical equivalence.
- (3) $\text{Pic}(X)_k = \text{Pic}(X) \otimes_{\mathbb{Z}} k$ for $k = \mathbb{Q}, \mathbb{R}$ and similarly for $N^1(X)_k$.
- (4) $\text{Amp}(X)$: the cone in $N^1(X)_{\mathbb{R}}$ spanned by ample divisors.
- (5) $\text{Nef}(X)$: the cone in $N^1(X)_{\mathbb{R}}$ spanned by nef divisors.
- (6) $\text{Mov}(X)$: the cone in $N^1(X)_{\mathbb{R}}$ spanned by movable divisors.
- (7) $\text{Big}(X)$: the cone in $N^1(X)_{\mathbb{R}}$ spanned by big divisors.
- (8) $\text{Eff}(X)$: the cone in $N^1(X)_{\mathbb{R}}$ spanned by effective divisors.
- (9) \overline{C} : the closure of the cone C .
- (10) $\text{Ud}(f)$: the undefined locus of a rational map $f : X \dashrightarrow Y$ on X .

Acknowledgements. I would like to thank my advisor Yongnam Lee, for his advice, encouragement and teaching. I also thank to Sung Rak Choi for helpful comments on limiting Okounkov bodies of pseudo-effective divisors. This work was supported by Basic Science Program through the National Research Foundation of Korea funded by the Korea government(MSIP)(No.2013006431).

2. OKOUNKOV BODIES AND MORI DREAM SPACES

In this section, we give some preliminaries which we need later on. We recall the definition, basic properties of the Okounkov bodies, and those of Mori dream spaces.

2.1. Contruction. In this subsection, we define the Okounkov bodies of big divisors on a projective variety.

Definition 2.1. Let X be a projective variety of dimension n . Consider a complete flag

$$X_\bullet : X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = \{\text{point}\}$$

of subvarieties of X , where $\text{codim}(X_i) = i$ and each X_i is smooth. We call this an admissible flag.

Remark 2.2. In this paper, we assume that all flags are admissible. A divisor means \mathbb{Z} -divisor unless otherwise stated.

Let X be a projective variety of dimension n , D a big divisor on X , and fix an admissible flag X_\bullet . Now, we define a valuation-like function

$$\nu := \nu_{X_\bullet, D} : H^0(X, \mathcal{O}_X(D)) \rightarrow \mathbb{Z}^n \cup \{\infty\}, s \mapsto \nu(s) = (\nu_1(s), \dots, \nu_n(s))$$

by the following way:

Let $0 \neq s \in H^0(X, \mathcal{O}_X(D))$ be a nonzero section. Define

$$\nu_1(s) := \text{ord}_{X_1}(s).$$

After choosing a local equation f for X_1 in X , s determines a section

$$\tilde{s}_1 := s \otimes f^{-\nu_1} \in H^0(X, \mathcal{O}_X(D - \nu_1 X_1)).$$

By restricting \tilde{s}_1 to X_1 , we obtain

$$s_1 \in H^0(X_1, \mathcal{O}_{X_1}(D - \nu_1 X_1)).$$

Then, $\nu_2(s) := \text{ord}_{X_2}(s_1)$. Once we have defined $\nu_i(s)$ for $i \leq n-1$, we define inductively by the same way s_i and ν_{i+1} . The values ν_i define the function ν as desired.

Note 2.3. The above $\nu = \nu_{X_\bullet}$ satisfies three valuation-like properties (this is why we call ν a valuation-like function):

- (1) $\nu_{X_\bullet}(s) = \infty$ if and only if $s = 0$.
- (2) $\nu_{X_\bullet}(s_1 + s_2) \geq \min\{\nu_{X_\bullet}(s_1), \nu_{X_\bullet}(s_2)\}$ by ordering \mathbb{Z}^n lexicographically, where $s_1, s_2 \in H^0(X, \mathcal{O}_X(D))$ are non-zero sections.
- (3) $\nu_{X_\bullet, D+E}(s \otimes t) = \nu_{X_\bullet, D}(s) + \nu_{X_\bullet, E}(t)$ for non-zero sections $s \in H^0(X, \mathcal{O}_X(D))$ and $t \in H^0(X, \mathcal{O}_X(E))$.

By using ν , we can define Okounkov bodies.

Definition 2.4. Let X, D be as above. Define

$$\Gamma(D)_m := \text{Im}(((H^0(X, \mathcal{O}_X(mD)) - \{0\}) \xrightarrow{\nu} \mathbb{Z}^n)$$

for all $m \in \mathbb{N}$. Define

$$\Sigma(\Gamma) = \text{closure of the convex hull of } \{(\Gamma(D)_m, m), m \in \mathbb{N}\}.$$

The Okounkov body of D with respect to the fixed flag X_\bullet is the compact convex set

$$\begin{aligned}\Delta_{X_\bullet}(D) &:= \Sigma(\Gamma) \cap (\mathbb{R}^n \times \{1\}) \\ &= \text{closed convex hull} \left(\bigcup_{m \geq 1} \frac{1}{m} \cdot \Gamma(D)_m \right) \subset \mathbb{R}^n.\end{aligned}$$

Remark 2.5. For a big \mathbb{Q} -divisor D , $\Delta_{X_\bullet}(D) := \frac{1}{m}(\Delta_{X_\bullet}(mD))$, where mD is an integral divisor. For definition of Okounkov bodies of big \mathbb{R} -divisors, see [3, Subsection 2.2].

By Remark 2.5, we can define limiting Okounkov body of a pseudo-effective divisor, which is a natural generalization of Definition 2.4.

Definition 2.6. Let D be a pseudo-effective divisor on X . The limiting Okounkov body $\Delta_{X_\bullet}^{\lim}(D)$ of D associated to the flag X_\bullet is defined as

$$\Delta_{X_\bullet}^{\lim}(D) := \lim_{\epsilon \rightarrow 0} \Delta_{X_\bullet}(D + \epsilon A) = \bigcap_{m \in \mathbb{N}} \Delta_{X_\bullet}(D + \frac{1}{m}A),$$

where A is a nef and big divisor on X . By the continuity of the Okounkov bodies, when D is big, $\Delta_{X_\bullet}^{\lim}(D) = \Delta_{X_\bullet}(D)$.

2.2. Restricted Okounkov bodies. We recall the definition of the restricted Okounkov bodies. Before going on, we first define the restricted complete linear series of a divisor.

Definition 2.7. (Restricted complete linear series) Let X be a projective variety and V a subvariety of X . Let D be a big divisor on V . Set

$$W_m = H^0(X|V, \mathcal{O}_X(mD)) := \text{Im}(H^0(X, \mathcal{O}_X(mD)) \xrightarrow{\text{rest}} H^0(V, \mathcal{O}_V(mD))),$$

where the map is the restriction map. We call such a graded linear series $\{W_m\}_{m \in \mathbb{Z}_{\geq 0}}$ on X the restricted complete linear series of D from X to V .

Now, we can define the restricted Okounkov bodies.

Definition 2.8. Let X be a projective variety and V a subvariety of X . Given an admissible flag X_\bullet , we call $\Delta_{X_\bullet|V}(D)$ the restricted Okounkov bodies of D to V if we take sections in the restricted complete linear series of D to V .

The importance of the restricted Okounkov bodies comes from the fact that it is closely related to the restriction of the Okounkov bodies of a divisor (Proposition 2.11).

Definition 2.9. [10, Definition 2.1.20, 10.3.2] Let X be a projective variety, and L a big divisor. The stable base locus of L is the algebraic set

$$\mathbf{B}(L) := \bigcap_{m \geq 1} \text{Bs}(|mL|).$$

The augmented base locus and restricted base locus of L are

$$\mathbf{B}_+(L) := \bigcap_A \mathbf{B}(L - A), \text{ and } \mathbf{B}_-(L) := \bigcup_A \mathbf{B}(L + A),$$

respectively, where the intersection and the union are taken over all ample divisors A .

Note 2.10. $\mathbf{B}_+(L)$ and $\mathbf{B}_-(L)$ are independent of the choice of A and sufficiently small $\epsilon > 0$.

Proposition 2.11. *Let X be a projective variety of dimension n and $X_\bullet : X \supset X_1 = E \supset X_2 \supset \cdots \supset X_n = \{\text{point}\}$ a given flag. Let D be a big divisor on X and suppose that $E \not\subseteq \mathbf{B}_+(D)$. For $t \in \mathbb{R}$ with $\Delta_{X_\bullet}(D)_{x_1=t} \neq \emptyset$,*

$$\Delta_{X_\bullet}(D)_{x_1=t} = \Delta_{X_\bullet|E}(D - tE).$$

Proposition 2.11 plays a central role in getting the Okounkov bodies of a big divisor on a surface. However, computing the restricted Okounkov body is another problem. In an ample divisor case, we can have an easy answer since the first cohomology vanishes by Serre vanishing. For details, see [7, Proposition 3.1].

Proposition 2.12. *Let X and X_\bullet be as in Proposition 2.11. Let A be an ample divisor on X . Then,*

$$\Delta_{X_\bullet|X_1}(A) = \Delta_{X_1}(A|_{X_1}).$$

2.3. Examples. In this subsection, we show some examples of the Okounkov bodies of big divisors on a curve, surface, and some special varieties.

Example 2.13. (Curve)

Let C be a smooth projective curve of genus g . Fix a point $p \in C$ and consider an admissible flag $C_\bullet : C \supseteq p$. Let D be a divisor on C . Consider a valuation

$$\nu_{C_\bullet} : H^0(C, \mathcal{O}_C(D)) \rightarrow \mathbb{Z}.$$

In this case, it is just $\text{ord}_p(s)$ for $s \in H^0(C, \mathcal{O}_C(D))$. Let $d := \deg_C(D)$. Then, it is clear that $\Delta_{C_\bullet}(D) \subseteq [0, d]$. Now, we show that the reverse inclusion holds. For $m \gg 0$ such that $md \geq 2g$, $|mD|$ is base point free. Thus, $\text{ord}_p(s) = 0$ for some $s \in H^0(C, \mathcal{O}_C(D))$. Moreover, for such $m \gg 0$, by Riemann-Roch, $h^0(mD) = md + 1 - g$. It is well-known that

$$\text{the number of points in } \nu_{C_\bullet}(H^0(mD)) = h^0(mD) = md + 1 - g$$

for all $m \gg 0$. So convex hull of $\nu_{C_\bullet}(H^0(mD))$ contains $[0, md - g]$ for all $m \gg 0$: i.e., $[0, d - \frac{g}{m}] \subseteq \Delta_{C_\bullet}(D)$ for all $m \gg 0$. Therefore, $\Delta_{C_\bullet}(D) = [0, d]$.

As in Example 2.13, the Okounkov bodies of a divisor in a curve is easy to obtain. However, even in surface cases, it becomes much more complicated.

Example 2.14. (Surface)

We refer the readers to [11, Section 6] for more details. Let X be a surface and $X_\bullet : X \supset C \supset x$ an admissible flag. The main idea is the following lemma.

Lemma 2.15. *Let D be a big \mathbb{Q} -divisor on a surface X with a Zariski decomposition $D = P + N$. Assume that $C \not\subseteq \mathbf{B}_+(D)$. Let*

$$\alpha(D) = \text{ord}_x(N|_C), \quad \beta(D) = \text{ord}_x(N|_C) + (C.P).$$

Then, the restricted Okounkov body of D is

$$\Delta_{X_\bullet|C}(D) = [\alpha(D), \beta(D)].$$

Before going on, we need to define the asymptotic valuation $\text{ord}_V(\|D\|)$. For more details, see [3, Subsection 2.1].

Definition 2.16. Let X be a projective variety and $V \subset X$ be an irreducible subvariety of X . For a big divisor D on X , define the asymptotic valuation of D on V as

$$\text{ord}_V(\|D\|) := \inf\{\text{ord}_V(D') \mid D \equiv D' \geq 0\}.$$

Now, $\mu := \sup\{s > 0 \mid D - sC \text{ is big}\}$. Then, we obtain the following proposition by using the above lemma with the linearity property of Zariski chambers in [2]. See [11, Theorem 6.4] for a proof.

Proposition 2.17. *Let X , X_\bullet , and D be as above. Then,*

$$\begin{aligned} \Delta_{X_\bullet}(D) = \{(x_1, x_2) \in \mathbb{R}^2 \mid & \text{ord}_C(\|D\|) \leq x_1 \leq \mu, \\ & \text{ord}_x(N_{D-x_1C}|_C) \leq x_2 \leq \text{ord}_x(N_{D-x_1C}|_C) + (C \cdot P_{D-x_1C})\}, \end{aligned}$$

where $\text{ord}_C(\|D\|)$ is the asymptotic valuation of D on C , $D - x_1C = P_{D-x_1C} + N_{D-x_1C}$ is the Zariski decomposition, and $\text{ord}_x(N_{D-x_1C}|_C)$ is the order of $N_{D-x_1C}|_C$ at x .

Remark 2.18. As you can see in the proof of [11, Theorem 6.4], the main idea is the existence of Zariski decompositions on a surface. However, it is not the case in higher dimensions. Thus, it is difficult to know the general expressions of Okounkov bodies in higher dimensional cases.

The following example is the simple case in higher dimensions. For notations, see Notation 1.4.

Example 2.19. (Threefold X with $\overline{\text{Eff}}(X) = \text{Nef}(X)$)

Let X be a threefold with $\overline{\text{Eff}}(X) = \text{Nef}(X)$. Then, the big cone and the ample cone coincide. Let $X_\bullet : X \supset S \supset C \supset \{p\}$ be an admissible flag, where S satisfies $\overline{\text{Eff}}(S) = \text{Nef}(S)$. Then, [7, Corollary 3.2] says that for any big divisor D (in this case, D is ample),

$$\Delta_{X_\bullet}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq \mu, 0 \leq x_2 \leq \mu_{x_1}, 0 \leq x_3 \leq ((D - x_1S)|_S - x_2C \cdot C)\},$$

where $\mu = \sup\{t > 0 \mid D - tS \in \text{Amp}(X)\}$, $\mu_{x_1} = \sup\{\alpha > 0 \mid (D - x_1S)|_S - \alpha C \in \text{Amp}(X)\}$.

However, if $\overline{\text{Eff}}(S) = \text{Nef}(X)$ does not hold, then by using [7, Proposition 3.1] with Example 2.14, we can describe $\Delta_{X_\bullet}(D)$.

2.4. Mori dream spaces. In this subsection, we briefly recall some definitions and results related to Mori dream spaces.

Definition 2.20. Let X be a normal projective variety.

- (1) A small \mathbb{Q} -factorial modification (SQM) of X is a small birational map $\mu : X \dashrightarrow Y$ to another normal \mathbb{Q} -factorial projective variety Y .
- (2) A contracting birational map is a birational map surjective in codimension one.

Definition 2.21. A normal projective \mathbb{Q} -factorial variety X is called a Mori dream space (MDS) if the following conditions hold.

- (1) $\text{Pic}(X)_{\mathbb{Q}} = \text{N}^1(X)_{\mathbb{Q}}$.

- (2) $\text{Nef}(X)$ is the affine hull of finitely many semi-ample divisors.
- (3) There are finitely many small \mathbb{Q} -factorial modifications $f_i : X \dashrightarrow X_i$ such that each X_i satisfies 1), 2) and $\text{Mov}(X) = \bigcup (f_i^*(\text{Nef}(X_i)))$.

The following proposition is a basic fact relating to Mori dream spaces. See [5, Proposition 1.11] for details.

Proposition 2.22. *Let X be a Mori dream space.*

- (1) *The f_i in Definition 2.21-(3) are all the small \mathbb{Q} -factorial modifications of X . In particular, the identity of X must appear among them.*
- (2) *There are finitely many contracting birational maps of X , $g_i : X \dashrightarrow Y_i$, with Y_i a Mori dream space, such that*

$$\text{Eff}(X) = \bigcup_i (g_i^*(\text{Nef}(Y_i)) * \text{Ex}(g_i))$$

gives a decomposition of $\text{Eff}(X)$ into closed rational polyhedral subcones (which we call them Mori chambers) with disjoint interiors, where $\text{Ex}(g_i)$ denotes the cone spanned by the irreducible components of the exceptional locus of g_i and $$ denotes the join.*

Now, we recall definitions of a Zariski decomposition and related propositions of Mori dream spaces using Proposition 2.22.

Definition 2.23. Let X be a normal projective variety and D a pseudo-effective \mathbb{Q} -divisor on X . A Zariski decomposition of D consists of a nef \mathbb{Q} -divisor P and an effective \mathbb{Q} -divisor N such that $D = P + N$ and for all sufficiently divisible positive integer m with mD , mP integral, the natural map

$$H^0(X, \mathcal{O}_X(mP)) \rightarrow H^0(X, \mathcal{O}_X(mD))$$

is an isomorphism, where the map is the multiplication of a canonical section of $\mathcal{O}_X(mN)$.

Moreover, if X is a Mori dream space, we say that $D = P + N$ is a Zariski decomposition of D in terms of MDS if there exists a small \mathbb{Q} -factorial modification $f : X \dashrightarrow X'$ such that

$$f_*^c D = f_*^c P + f_*^c N$$

is a Zariski decomposition in the above sense, where f_*^c is the cycle pushforward of codimension 1 cycles.

One of the most important features of a divisor on a smooth projective surface is that it has a Zariski decomposition. In general, in higher dimensions, we cannot say that. However, the following proposition says that in Mori dream spaces, we can say the similar one. See [12, Proposition 2.13] for details.

Proposition 2.24. *Let X be a Mori dream space. Consider the decomposition of $\text{Eff}(X)$ in Proposition 2.22. Then, for each chamber C , there exists a small \mathbb{Q} -factorial modification $f_i : X \dashrightarrow X_i$ of X and two \mathbb{Q} -linear maps*

$$P, N : C \rightarrow \text{Eff}(X)$$

such that for any integral $D \in C$, $D = P(D) + N(D)$ gives a Zariski decomposition of D as a divisor on X_i .

Conversely, let X be a \mathbb{Q} -factorial normal projective variety such that $\text{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$. Assume that $\text{Eff}(X)$ is decomposed into finitely many chambers C on each of which there exists \mathbb{Q} -linear Zariski decompositions in the above sense, with positive parts semi-ample on some small \mathbb{Q} -factorial modifications of X . Then, X is a Mori dream space.

Remark 2.25. We call C in Proposition 2.24 as Mori chamber and $\text{Eff}(X) = \bigcup C$ a Mori chamber decomposition.

3. SLICES OF OKOUNKOV BODIES OF MORI DREAM SPACES

When we deal with the Okounkov bodies in higher dimensions, the main obstruction is that we cannot ensure that every divisor has a Zariski decomposition. It does not happen in general since nef cone and movable cone are not the same in higher dimensions. This poses the difficulty in finding the restricted Okounkov bodies, which makes it hard to find the Okounkov bodies of a given divisor.

However, in Mori dream spaces, although not all pseudo-effective divisors may have Zariski decompositions in the usual sense, Proposition 2.24 implies that all the divisors have a decomposition that is similar with a Zariski decomposition. In this section, we analyze all the slices of Okounkov bodies of Mori dream spaces. As a byproduct, we obtain the description of Okounkov bodies of Mori dream threefolds. Now, we start it by fixing some notations.

Note 3.1. In this paper, a divisor means an integral divisor. However, all the arguments in this section can be extended to \mathbb{Q} -divisors by the homogeneity of the Okounkov body. Moreover, by the continuity of the Okounkov body, we can extend them to \mathbb{R} -divisors. We leave the details to the reader.

Notation 3.2. Let X be a Mori dream space of dimension n , $f_i : X \dashrightarrow X_i$ SQMs of X , D a big divisor on X , and $X_{\bullet} : X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}$ an admissible flag with $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, where I_D is defined on Note 3.4. Finally, let $\text{Eff}(X) = \bigcup_{i=1}^r C_i$ be a Mori chamber decomposition of $\text{Eff}(X)$.

Lemma 3.3. Let X be a normal, projective variety of dimension n and D a pseudo-effective divisor on X . Let $\mu : X' \rightarrow X$ be a projective birational morphism. Then,

$$\Delta_{X_{\bullet}}^{\lim}(D) = \Delta_{X'_{\bullet}}^{\lim}(\mu^*D)$$

for any admissible flag $X_{\bullet} : X_0 = X \supset X_1 \supset \cdots \supset X_n$ with X_n not contained in the center of the exceptional locus of μ , and the admissible flag X'_{\bullet} induced by the strict transform of X_{\bullet} .

Proof. Let $\mu : X' \rightarrow X$ be a projective birational morphism. Since X_n is not contained in the center of the exceptional locus of μ , we can define X'_{\bullet} using strict transform of X_{\bullet} . First, let D be a big divisor. Since X is normal, $H^0(X, \mathcal{O}_X(mD)) = H^0(X', \mathcal{O}_{X'}(\mu^*mD))$ for all $m \in \mathbb{N}$. Therefore, we conclude that $\Delta_{X_{\bullet}}(D) = \Delta_{X'_{\bullet}}(\mu^*D)$.

In general, let D be a pseudo-effective divisor, A a nef and big divisor on X . Since $D + \frac{1}{m}A$ is big for all $m \in \mathbb{N}$,

$$\Delta_{X_\bullet}^{\lim}(D) = \bigcap_{m \in \mathbb{N}} \Delta_{X_\bullet}(D + \frac{1}{m}A) = \bigcap_{m \in \mathbb{N}} \Delta_{X'_\bullet}(\mu^*D + \frac{1}{m}\mu^*A) = \Delta_{X'_\bullet}^{\lim}(\mu^*D).$$

,

□

Note 3.4. (Set-up and notation)

Notation is as in Notation 3.2. Consider $\{t \in \mathbb{R}_{\geq 0} \mid D - tY_1 \in \bar{C}_i\}$, denoted by $[\alpha_i, \beta_i]$. (Note : It is just one closed interval since each Mori chamber is convex.) Consider the elimination of indeterminacy of f_i ,

$$\begin{array}{ccc} \tilde{X}_i & & \\ \downarrow \phi_i & \searrow \tilde{f}_i & \\ X & \xrightarrow{f_i} & X_i \end{array}$$

Now, let $D_t := D - tY_1$ for $t \in \mathbb{R}_{\geq 0}$, and $\tilde{X}_{i\bullet} : \tilde{X}_i = \tilde{Y}_0^i \supset \tilde{Y}_1^i \supset \cdots \supset \tilde{Y}_n^i$ a flag induced by strict transforms of X_\bullet using ϕ_i (It makes sense since $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$ holds). Let $\mu = \sup\{t > 0 \mid D - tY_1 \in \text{Big}(X)\}$. Let $D_t = P_{D_t} + N_{D_t}$ be its Zariski decomposition in terms of MDS, and $I_D := \{i \mid D_t \in C_i \text{ for some } t \in [\text{ord}_{Y_1}(\|D\|), \mu]\}$, where $\mu = \sup\{t > 0 \mid D - tY_1 \in \text{Big}(X)\}$.

Before going on, we need some lemmas.

Lemma 3.5. *Notation is as in Notation 3.2 and Note 3.4. Then, $\Delta_{X_\bullet|Y_1}(D_t) = \Delta_{\tilde{X}_{i\bullet}|Y_1^i}(\phi_i^*D_t)$ for all $t \in [\text{ord}_{Y_1}(\|D\|), \mu]$.*

Proof. By Lemma 3.3, we have $\Delta_{X_\bullet}(D_t) = \Delta_{\tilde{X}_{i\bullet}}(\phi_i^*D_t)$. Therefore, by [11, Theorem 4.24],

$$\Delta_{X_\bullet|Y_1}(D_t) = \Delta_{X_\bullet}(D_t)_{x_1=0} = \Delta_{\tilde{X}_{i\bullet}}(\phi_i^*D_t)_{x_1=0} = \Delta_{\tilde{X}_{i\bullet}|Y_1^i}(\phi_i^*D_t).$$

□

Lemma 3.6. *Notation is as in Notation 3.2 and Note 3.4. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), \dots, l_i^{n-1}(t))$ defined on each $[\alpha_i, \beta_i]$ such that*

$$\Delta_{\tilde{X}_{i\bullet}|Y_1^i}(\tilde{f}_i^*(f_{i*}^c D_t)) = \Delta_{\tilde{X}_{i\bullet}|Y_1^i}(\tilde{f}_i^*(f_{i*}^c P_{D_t})) + l_i(t).$$

Proof. Note that X_i is normal. Also, since \tilde{f}_i is birational and proper, by Zariski main theorem and Stein factorization, $f_{i*}\mathcal{O}_{\tilde{X}_i} = \mathcal{O}_{X_i}$. Now, consider the following isomorphisms.

$$H^0(\tilde{X}_i, \tilde{f}_i^*(f_{i*}^c P_{D_t})) \xrightarrow[\cong]{g_1} H^0(X_i, f_{i*}^c P_{D_t}) \xrightarrow[\cong]{g_2} H^0(X_i, f_{i*}^c D_t) \xrightarrow[\cong]{g_3} H^0(\tilde{X}_i, \tilde{f}_i^*(f_{i*}^c D_t)).$$

The middle isomorphism (g_2) comes from the Zariski decomposition property of a Mori dream space, and the first (g_1) and the last (g_3) ones come from the projection formula with $\tilde{f}_{i*}\mathcal{O}_{\tilde{X}_i} = \mathcal{O}_{X_i}$: i.e.,

$$f_{i*}^c D_t = \tilde{f}_{i*}\mathcal{O}_{\tilde{X}_i} \otimes f_{i*}^c D_t \rightarrow \tilde{f}_{i*}(\mathcal{O}_{\tilde{X}_i} \otimes \tilde{f}_i^*(f_{i*}^c D_t)) = \tilde{f}_{i*}\tilde{f}_i^*(f_{i*}^c D_t),$$

which is just the pull-back section map when we take global sections. Thus,

$$\tilde{f}_i^* s \xrightarrow{g_1} s \xrightarrow{g_2} s \otimes s_{N_{D_t}} \xrightarrow{g_3} \tilde{f}_i^* s \otimes \tilde{f}_i^* s_{N_{D_t}},$$

where $s_{N_{D_t}} \in H^0(X_i, f_{i*}^c N_{D_t})$ is the canonical section. It also holds when we consider $m\tilde{f}_i^*(f_{i*}^c D_t)$, $m\tilde{f}_{i*}^c D_t$, $m\tilde{f}_{i*}^c P_{D_t}$, and $m\tilde{f}_i^*(f_{i*}^c P_{D_t})$ by taking $s_{N_{D_t}}^{\otimes m} \in H^0(X_i, m\tilde{f}_{i*}^c N_{D_t})$ for $m \geq 1$. Therefore, since pull-back functor commutes with \otimes , for $m \geq 1$,

$$\begin{aligned} H^0(\tilde{X}_i | \tilde{Y}_1^i, m\tilde{f}_i^*(f_{i*}^c P_{D_t})) &= \{(\tilde{f}_i^* s)|_{\tilde{Y}_1^i} \mid s \in H^0(X_i, m\tilde{f}_{i*}^c P_{D_t})\}, \\ H^0(\tilde{X}_i | \tilde{Y}_1^i, m\tilde{f}_i^*(f_{i*}^c D_t)) &= \{(\tilde{f}_i^* s)|_{\tilde{Y}_1^i} \otimes (\tilde{f}_i^* s_{N_{D_t}})^{\otimes m}|_{\tilde{Y}_1^i} \mid s \in H^0(X_i, m\tilde{f}_{i*}^c P_{D_t})\}. \end{aligned}$$

Now, let $l_i(t) = \nu_{\tilde{Y}_1^i}((\tilde{f}_i^* s_{N_{D_t}})|_{\tilde{Y}_1^i})$. Since $\nu_{\tilde{Y}_1^i}(s_1 \otimes s_2) = \nu_{\tilde{Y}_1^i}(s_1) + \nu_{\tilde{Y}_1^i}(s_2)$, we obtain the desired result except that $l_i(t)$ is linear on $[\alpha_i, \beta_i]$. However, the linearity comes from the linearity of N_{D_t} in each Mori chamber (see Proposition 2.24) and the well-known fact that $s_D = s_{D_1} \otimes s_{D_2}$, where s_D, s_{D_i} are the canonical sections of effective divisors D and D_i such that $D = D_1 + D_2$, which proves the lemma. \square

Remark 3.7. In fact, we do not have to assume ‘ $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$ ’ in Lemma 3.6. However, this assumption is essential for Proposition 3.8.

Proposition 3.8. *Notation is as in Notation 3.2 and Note 3.4. Then,*

$$\Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D_t) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^*(f_{i*}^c D_t))$$

for each $i = 1, \dots, r$.

Proof. Note that $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$. Let $U \subset X$ and $U_i \subset X_i$ be isomorphic loci of ϕ_i , f_i , and \tilde{f}_i . Now, consider strict transforms of $\tilde{f}_i^*(f_{i*}^c D_t)$ and $\phi_i^* D_t$, denoted by $\tilde{f}_{i*}^{-1}(f_{i*}^c D_t)$ and $\phi_{i*}^{-1} D_t$, respectively. Note that

$$\begin{aligned} \tilde{f}_{i*}^{-1}(f_{i*}^c D_t) &= \overline{\tilde{f}_i^{-1}(f_i(D_t \cap U) \cap U_i)} = \overline{\phi_i^{-1} \circ f_i^{-1}(f_i(D_t \cap U) \cap U_i)} \\ &= \overline{\phi_i^{-1} \circ f_i^{-1}(f_i(D_t \cap U) \cap U_i)} = \overline{\phi_i^{-1}(D_t \cap U)}. \end{aligned}$$

Also, since $\phi_{i*}^{-1} D_t = \overline{\phi_i^{-1}(D_t \cap U)}$, we obtain that their strict transforms are the same. By the assumption, $\tilde{f}_i^*(f_{i*}^c D_t)|_{\tilde{Y}_1^i} = \phi_i^* D_t|_{\tilde{Y}_1^i}$. Thus, we obtain the following diagram

$$\begin{array}{ccc} H^0(\tilde{X}_i, \phi_i^* D_t) & \xrightarrow{\psi_1} & H^0(\tilde{Y}_1^i, \phi_i^* D_t|_{\tilde{Y}_1^i}) \\ & & \parallel \\ H^0(\tilde{X}_i, \tilde{f}_i^*(f_{i*}^c D_t)) & \xrightarrow{\psi_2} & H^0(\tilde{Y}_1^i, \tilde{f}_i^*(f_{i*}^c D_t)|_{\tilde{Y}_1^i}), \end{array}$$

where ψ_1 and ψ_2 are restriction maps. Since the strict transforms of $\tilde{f}_i^*(f_{i*}^c D_t)$ and $\phi_i^* D_t$ are exactly the same, and $\tilde{Y}_1^i \cap E = \emptyset$ for any exceptional divisor E by the assumption $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, the images of ψ_1 and ψ_2 are the same. Thus, we obtain $\Delta_{\tilde{X}_{i\bullet} | \tilde{Y}_1^i}(\phi_i^* D_t) = \Delta_{\tilde{X}_{i\bullet} | \tilde{Y}_1^i}(\tilde{f}_i^*(f_{i*}^c D_t))$ for each $t =$

$t_i \in [\alpha_i, \beta_i]$. Then, for any $t \in [\text{ord}_{\tilde{Y}_1}(\|\phi_i^* D\|), \mu] = [\text{ord}_{\tilde{Y}_1}(\|\tilde{f}_i^*(f_{i*}^c D)\|), \mu]$, we obtain that

$$\begin{aligned} \Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D)_{x_1=t} &= \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\phi_i^* D - t\tilde{Y}_1^i) = \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\phi_i^*(D - tY_1)) \\ &= \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\tilde{f}_i^* f_{i*}^c(D - tY_1)) = \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\tilde{f}_i^* f_{i*}^c D - t\tilde{Y}_1^i) \\ &= \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c D)_{x_1=t}. \end{aligned}$$

Thus, $\Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c D)$. \square

By Proposition 3.8, we can obtain the following corollary, which is a special case of [9, Theorem C].

Corollary 3.9. *Let X be a Mori dream space of dimension n , $X_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_n$ an admissible flag, and D a big divisor on X such that $D \in C_i$, where C_i is a Mori chamber. Assume that $Y_1 \cap \text{Ud}(f_i) = \emptyset$, where f_i is a SQM of X corresponding to C_i . Then,*

$$\Delta_{X_\bullet}(D) = \Delta_{X_\bullet}(P_D) + a$$

for some vector $a \in \mathbb{R}^n$, where P_D is the positive part of the Zariski decomposition of D in terms of MDS.

Proof. We use notations in Notation 3.2 and Note 3.4. First, assume that D , P_D , and N_D are all integral. By Proposition 3.8, $\Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c D)$. Let $a := \nu_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* s_{N_D}) \in \mathbb{R}^n$ for the canonical section $s_{N_D} \in H^0(X_i, f_{i*}^c N_D)$. By using the same argument in the proof of Lemma 3.6, we can obtain $\Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c D) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c P_D) + a$. Then, by Lemma 3.3, $\Delta_{X_\bullet}(D) = \Delta_{\tilde{X}_{i\bullet}}(\phi_i^* D) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c D) = \Delta_{\tilde{X}_{i\bullet}}(\tilde{f}_i^* f_{i*}^c P_D) + a = \Delta_{\tilde{X}_{i\bullet}}(\phi_i^* P_D) + a = \Delta_{X_\bullet}(P_D) + a$.

In general, assume that D , P_D , and N_D are \mathbb{Q} -divisors. Choose $m \gg 0$ such that mD , mP_D , and mN_D are all integral. Then, by using the same argument as in the integral case to mD , mP_D , and mN_D , we obtain the desired result. \square

Now, we are ready to prove our main theorem.

Theorem 3.10. *Let X , X_\bullet , D , I_D , and $[\alpha_i, \beta_i]$ be as in Notation 3.2 and Note 3.4. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), \dots, l_i^{n-1}(t))$ defined on each $[\alpha_i, \beta_i]$ such that*

$$\Delta_{X_\bullet}(D)_{x_1=t} = \Delta_{Y_1\bullet}(P_{D_t}|_{Y_1}) + l_i(t)$$

for all $t = t_i \in [\alpha_i, \beta_i]$.

Proof. Let f_i be SQMs of X . Fix $t = t_i \in [\alpha_i, \beta_i]$. By Lemma 3.6 and Proposition 3.8, we obtain

$$\begin{aligned} \Delta_{X_\bullet}(D)_{x_1=t} &= \Delta_{X_\bullet|Y_1}(D_t) = \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\phi_i^* D_t) = \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\tilde{f}_i^*(f_{i*}^c D_t)) \\ &= \Delta_{\tilde{X}_{i\bullet}|\tilde{Y}_1^i}(\tilde{f}_i^*(f_{i*}^c P_{D_t})) + l_i(t) = \Delta_{\tilde{Y}_1\bullet}(\tilde{f}_i^*(f_{i*}^c P_{D_t})|_{\tilde{Y}_1^i}) + l_i(t), \end{aligned}$$

where the second equality holds by Lemma 3.5, and the last equality comes from the continuity of the restricted Okounkov bodies and Proposition 2.12 since $\tilde{f}_i^*(f_{i*}^c P_{D_t})$ is nef. Now, consider the following commutative diagram.

$$\begin{array}{ccc}
\tilde{Y}_1 & \xrightarrow{i_{\tilde{Y}_1^i}} & \tilde{X}_i \\
\phi_i|_{\tilde{Y}_1^i} \downarrow & & \downarrow \phi_i \\
Y_1 & \xrightarrow{i_{Y_1}} & X
\end{array}$$

Since $Y_1 \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, note that

$$\begin{aligned}
(\tilde{f}_i^*(f_{i*}^c P_{D_t}))|_{\tilde{Y}_1^i} &= i_{\tilde{Y}_1^i}^*(\tilde{f}_i^*(f_{i*}^c P_{D_t})) = i_{\tilde{Y}_1^i}^*(\phi_i^* f_i^*(f_{i*}^c P_{D_t})) \\
&= (\phi_i|_{\tilde{Y}_1^i})^*(f_i^*(f_{i*}^c P_{D_t}))|_{Y_1} = (\phi_i|_{\tilde{Y}_1^i})^*(P_{D_t}|_{Y_1}).
\end{aligned}$$

Thus, $\Delta_{\tilde{Y}_1^i}(\tilde{f}_i^*(f_{i*}^c P_{D_t})|_{\tilde{Y}_1^i}) = \Delta_{\tilde{Y}_1^i}((\phi_i|_{\tilde{Y}_1^i})^*(P_{D_t}|_{Y_1})) = \Delta_{Y_1}(\phi_i^*(P_{D_t}|_{Y_1}))$. Therefore, $\Delta_{X_\bullet}(D)_{x_1=t} = \Delta_{Y_1}(P_{D_t}|_{Y_1}) + l_i(t)$. \square

Note 3.11. We give an expression of a Zariski decomposition of big divisors in terms of MDS (see the proof of [12, Proposition 2.13] for details). Let $g : X \rightarrow Y$ be contracting birational maps in Proposition 2.22-(2). Then, for any big divisor D on X , $P_D = g^* g_*^c D$ and $N_D = D - P_D$.

Remark 3.12. In this remark, we observe the meaning of Theorem 3.10 and its extension to pseudo-effective divisors.

- (1) Theorem 3.10 says that the problem on the descriptions of the Okounkov bodies of big divisors on X associated to X_\bullet is reduced to that on Y_1 associated to $Y_{1\bullet}$.
- (2) Theorem 3.10 can be extended to the limiting Okounkov bodies of pseudo-effective divisors naturally. We leave the details to the reader.
- (3) The first condition $(\text{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}})$ in Definition 2.21 is not essential for Theorem 3.10. In fact, Theorem 3.10 holds for any variety such that every big divisor has a decomposition in Proposition 2.24.

Remark 3.13. (Computations of Mori chamber decomposition) For Theorem 3.10, we need to know what Mori chamber decomposition of a given $\overline{\text{Eff}}(X)$ is. For computations of Mori chamber decomposition, see [6, Chapter 3].

As a byproduct of Theorem 3.10, we obtain the descriptions of the Okounkov bodies of big divisors on Mori dream threefolds.

Corollary 3.14. *Let X be a Mori dream threefold, and $X_\bullet : X \supset S \supset C \supset \{p\}$ an admissible flag with $S \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$, where D , I_D , and $[\alpha_i, \beta_i]$ are as in Notation 3.2 and Note 3.4. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), l_i^2(t))$ defined on each $[\alpha_i, \beta_i]$ such that*

$$\begin{aligned}
\Delta_{X_\bullet}(D) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{ord}_S(\|D\|) \leq x_1 \leq \mu, \text{ for each } x_1 = t_i \in [\alpha_i, \beta_i], \\
&\quad l_i^1(t_i) \leq x_2 \leq \mu_{t_i} + l_i^1(t_i), \delta_{x_2}(t_i) \leq x_3 \leq \delta_{x_2}(t_i) + (P_{(P_{D_{t_i}}|_{S-x_2C})} \cdot C)\},
\end{aligned}$$

where $\mu = \sup\{t > 0 \mid D - tS \in \text{Big}(X)\}$, $\mu_t = \sup\{\alpha > 0 \mid P_{D_t}|_S - \alpha C \in \text{Big}(S)\}$, $P_{D_{t_i}}|_S - x_2 C = P_{(P_{D_{t_i}}|_S - x_2 C)} + N_{(P_{D_{t_i}}|_S - x_2 C)}$ is the Zariski decomposition in the usual sense, and $\delta_{x_2}(t_i) = \text{ord}_p(N_{(P_{D_{t_i}}|_S - x_2 C)}|_C) + l_i^2(t_i)$ is a linear function on each $[\alpha_i, \beta_i]$ for fixed $x_2 \in [l_i^1(t_i), \mu_{t_i} + l_i^1(t_i)]$.

Proof. By Theorem 3.10, $\Delta_{X_\bullet}(D)_{x_1=t} = \Delta_{S_\bullet}(P_{D_t}|_S) + l_i(t)$, where $D_t := D - tS$. We are reduced to the surface case. By Example 2.14,

$$\Delta_{S_\bullet}(P_{D_t}|_S) = \{(x_2, x_3) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq \mu_t, \\ \text{ord}_p(N_{(P_{D_t}|_S - x_2 C)}|_C) \leq x_3 \leq \text{ord}_p(N_{(P_{D_t}|_S - x_2 C)}|_C) + (P_{(P_{D_t}|_S - x_2 C)} \cdot C)\},$$

where $\mu_t := \sup\{\alpha > 0 \mid P_{D_t}|_S - \alpha C \in \text{Big}(S)\}$ and $N_{(P_{D_t}|_S - x_2 C)}$ is the negative part of $P_{D_t}|_S - x_2 C$. Therefore, by summarizing them, we obtain the desired result. \square

Caution 3.15. The description of $\Delta_{X_\bullet}(D)$ in Corollary 3.14 may not hold without the assumption ' $S \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset$ '. For a counterexample, see Example 3.18.

Corollary 3.14 can be extended to the limiting Okounkov bodies of pseudo-effective divisors. Since the proof is similar to that of Corollary 3.14, we omit it here. For descriptions of the limiting Okounkov bodies of a surface, see [4, Theorem 4.5].

Corollary 3.16. Let X , X_\bullet , and $[\alpha_i, \beta_i]$ be as in Corollary 3.14. Let D be a pseudo-effective divisor, and $I_D := \{i \mid D_t \in C_i \text{ for some } t \in [\text{ord}_S(\|D\|), \mu]\}$, where $\mu = \sup\{t > 0 \mid D - tS \in \text{Eff}(X)\}$. Then, for each $i = 1, \dots, r$, there exist the linear function $l_i(t) = (l_i^1(t), l_i^2(t))$ defined on each $[\alpha_i, \beta_i]$ such that

$$\Delta_{X_\bullet}^{\lim}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{ord}_S(\|D\|) \leq x_1 \leq \mu, \text{ for each } x_1 = t_i \in [\alpha_i, \beta_i], \\ l_i^1(t_i) \leq x_2 \leq \mu_{t_i} + l_i^1(t_i), \delta_{x_2}(t_i) \leq x_3 \leq \delta_{x_2}(t_i) + (P_{(P_{D_{t_i}}|_S - x_2 C)} \cdot C)\},$$

where $\mu_t = \sup\{\alpha > 0 \mid P_{D_t}|_S - \alpha C \in \overline{\text{Eff}}(S)\}$, $P_{D_{t_i}}|_S - x_2 C = P_{(P_{D_{t_i}}|_S - x_2 C)} + N_{(P_{D_{t_i}}|_S - x_2 C)}$ is the Zariski decomposition in the usual sense, and $\delta_{x_2}(t_i) = \text{ord}_p(N_{(P_{D_{t_i}}|_S - x_2 C)}|_C) + l_i^2(t_i)$ is a linear function on each $[\alpha_i, \beta_i]$ for fixed $x_2 \in [l_i^1(t_i), \mu_{t_i} + l_i^1(t_i)]$.

Remark 3.17. Note that [7, Corollary 3.2] describes the Okounkov bodies of big divisors on X , where X is a smooth projective threefold such that $\overline{\text{Eff}}(X) = \text{Nef}(X)$. In this case, it is clear that every effective divisor has a decomposition in Proposition 2.24, and that there are no undefined loci of SQMs of X (so that we can choose any admissible flag). Therefore, by Remark 3.12-(3), Corollary 3.14 and 3.16 can be seen as a generalization of [7, Corollary 3.2].

Now, we see some examples of Corollary 3.14 and 3.16.

Example 3.18. Let X be the blowing-up of \mathbb{P}^3 at two points p_1 and p_2 with exceptional divisors E_1 and E_2 . By [1, Example 5.5], X is a Mori dream space. Let $\phi : X \rightarrow \mathbb{P}^3$ be the blowing-up map. Let $X_\bullet : X \supset S = E_1 \cong \mathbb{P}^2 \supset C \supset \{x\}$ be an admissible flag, where C is any curve in E_1 and x is any point in C . Let d be the degree of C and L a line in E_1 .

Note that $C = dL$. For notational convenience, let $D_t := D - tS$ for some pseudo-effective divisor D with $t \in [\text{ord}_S(\|D\|), \mu]$.

Now, let H be a hyperplane in \mathbb{P}^3 passing through p_1 and p_2 . Furthermore, let $H_1 := \phi^*H - E_1$, $H_2 := \phi^*H - E_2$, and H_{12} strict transform of plane passing through both p_1 and p_2 . For descriptions of the (limiting) Okounkov bodies, we need to know Mori chamber decomposition of $\text{Eff}(X)$. By [1, Example 5.5], X has two Mori chambers, and denote nef parts of two Mori chambers by N and N' , and each N and N' is generated by ϕ^*H , H_1 , H_2 and H_{12} , H_1 , H_2 , respectively. Moreover, let $\textcircled{1} := \langle \phi^*H, E_1, E_2 \rangle$, and $\textcircled{2} := \langle \phi^*H, H_1, H_2 \rangle$, where $\langle A, B, C \rangle$ denotes the closed convex cone generated by A , B , and C (FIGURE 1). First, consider a pseudo-effective divisor D satisfying the condition on Corollary 3.16 (or Corollary 3.14).

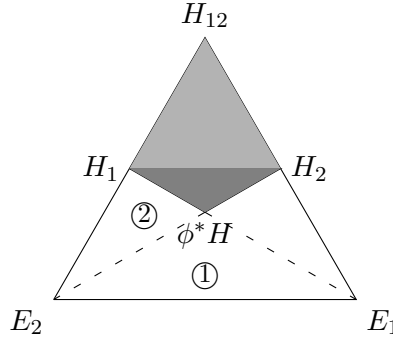


FIGURE 1. Mori chamber decomposition of $\overline{\text{Eff}}(X)$ (upper side)

Let D be a pseudo-effective divisor on X such that $D \in \textcircled{1}$. We may let $D = a\phi^*H + bE_2 + cE_1$ with $a, b, c \geq 0$ with at least one nonzero a, c (clearly, $\Delta_{X_\bullet}^{\lim}(bE_2) = \{(0, 0, 0)\}$). Note that $c \leq x_1 \leq a + c$. Also, for all $t \in [c, a + c]$, $D - tE_1 = aH_1 + (a + c - t)E_1 + bE_2$, so $P_{D_t} = aH_1 + (a + c - t)E_1$ and $N_{D_t} = bE_2$. So $P_{D_{x_1}}|_{E_1} = \frac{1}{d}(x_1 - c)C$, $\mu_{x_1} = \frac{1}{d}(x_1 - c)$ with $l_i(x_1) = (0, 0)$ for all $x_1 \in [c, a + c]$. Therefore, we obtain

$$\Delta_{X_\bullet}^{\lim}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid c \leq x_1 \leq a + c, 0 \leq x_2 \leq \frac{1}{d}(x_1 - c), \\ 0 \leq x_3 \leq dx_1 - d^2x_2 - dc\}.$$

Next, let D be a pseudo-effective divisor on X such that $D \in \textcircled{2}$. We may let $D = aH_1 + bE_2 + c\phi^*H$ with $a, b, c \geq 0$ with at least one nonzero a, c . Note that $0 \leq x_1 \leq c$. Also, for all $t \in [0, c]$, $D - tE_1 = aH_1 + bE_2 + cH_1 + (c - t)E_1 = (a + c)H_1 + (c - t)E_1 + bE_2$. So $P_{D_{x_1}} = (a + c)H_1 + (c - x_1)E_1$, $P_{D_{x_1}}|_{E_1} = \frac{1}{d}(x_1 + a)C$, $\mu_{x_1} = \frac{1}{d}(x_1 + a)$, and $N_{D_{x_1}} = bE_2$: i.e., $l_i(x_1) = (0, 0)$ for all $x_1 \in [0, c]$. Therefore, we obtain

$$\Delta_{X_\bullet}^{\lim}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq c, 0 \leq x_2 \leq \frac{1}{d}(x_1 + a), \\ 0 \leq x_3 \leq dx_1 - d^2x_2 + da\}.$$

Now, let us see an example of Caution 3.15. Let g be a non-trivial SQM of X . Since $E_1 \cap \text{Ud}(g) \neq \emptyset$ ([1, Example 5.5]), any ample divisor does not satisfy the assumption $(S \cap (\cup_{i \in I_D} \text{Ud}(f_i)) = \emptyset)$ in Corollary 3.14. Suppose

that all ample divisors satisfy the description of Corollary 3.14. Then, for such D , we can easily obtain that

$$\Delta_{X_\bullet}(D) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq \mu, 0 \leq x_2 \leq \frac{1}{d}x_1 + \frac{1}{d^2}(D.C), \\ 0 \leq x_3 \leq dx_1 - d^2x_2 + (D.C)\}.$$

Fix an ample divisor A , and consider $H_2 + \epsilon A$ with $\epsilon > 0$. Clearly, $H_2 + \epsilon A$ is ample, so by definition, we obtain that

$$\Delta_{X_\bullet}^{\lim}(H_2) := \lim_{\epsilon \rightarrow 0} \Delta_{X_\bullet}(H_2 + \epsilon A) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq \frac{1}{d}x_1, \\ 0 \leq x_3 \leq dx_1 - d^2x_2\}.$$

This is a contradiction since $\Delta_{X_\bullet}^{\lim}(H_2)$ cannot be of full-dimensional.

Example 3.19. Let X be a \mathbb{Q} -factorial, normal hypersurface of any bidegree in $\mathbb{P}^2 \times \mathbb{P}^2$. Then, by [15, Theorem 1.1], X is a Mori dream threefold with $\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \text{Nef}(X)$. More precisely, let $H_i = pr_i^*(\mathcal{O}_{\mathbb{P}^2}(1))$ for $i = 1, 2$, where pr_i is the i -th projection. Then,

$$\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \text{Nef}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}H_2 \text{ (FIGURE 2)}.$$

In this case, since small \mathbb{Q} -factorial modification is only the identity of X ,

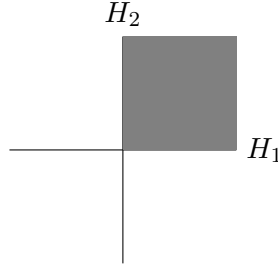


FIGURE 2. $\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \text{Nef}(X)$ in $N^1(X)_{\mathbb{R}}$

there is no undefined locus of SQM on X : we can choose any admissible flag of X . Thus, we can easily obtain the limiting Okounkov bodies of any pseudo-effective divisors on X for any admissible flags by using Corollary 3.16 as in Example 3.18. We leave the details to the reader.

Example 3.20. Let X be a general hypersurface of bidegree (d, e) in $\mathbb{P}^1 \times \mathbb{P}^3$ with $d \leq 3$. More precisely, let

$$f = x_0^d f_0 + x_0^{d-1} x_1 f_1 + \cdots + x_1^d f_d$$

be the defining bihomogeneous polynomial of bidegree (d, e) of X , where x_0, x_1 are coordinates on \mathbb{P}^1 and the f_i are homogeneous forms of degree e on \mathbb{P}^3 . Note that X is general in the sense that it is smooth and $(f_0 = \cdots = f_d = 0)$ is a smooth subvariety in \mathbb{P}^3 of codimension $d + 1$ ($(f_0 = \cdots = f_d = 0) = \emptyset$ if $d = 3$). As in Example 3.19, it is better to see [15, Theorem 1.1] for this example. Note that if d satisfies $1 \leq d \leq 3$ or $e = 1$, X is a Mori dream threefold. Now, consider the following cases:

(Case 1 : $d = 3$ or $e = 1$) In this case, $\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \text{Nef}(X)$. So there is only one SQM of X , which is the identity: we can take any pseudo-effective divisor and any admissible flag for Corollary 3.16. For descriptions of the limiting Okounkov bodies, as in Example 3.19, we leave the details to the reader.

(Case 2 : $d = 1$ with $e \geq 2$) In this case, by [15, Theorem 1.1],

$$\overline{\text{Eff}}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}E \text{ (FIGURE 3),}$$

$$\overline{\text{Mov}}(X) = \text{Nef}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}H_2,$$

where H_i is the pullback of hyperplanes by using i -th projection and $E = eH_2 - H_1$. More precisely, if we let $f = x_0f_0 + x_1f_1$ to be the defining

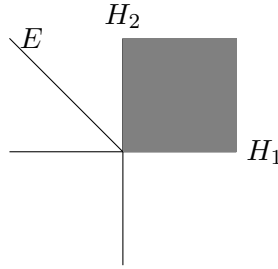


FIGURE 3. $\overline{\text{Eff}}(X)$ in $N^1(X)_{\mathbb{R}}$

bihomogeneous polynomial of X in $\mathbb{P}^1 \times \mathbb{P}^3$, then X can be viewed as the blowing-up of \mathbb{P}^3 along a curve $C'_0 := (f_0 = f_1 = 0)$, where the blowing-up map π is the second projection of $\mathbb{P}^1 \times \mathbb{P}^3$ and E is the exceptional divisor of this blowing-up. Also, since $\overline{\text{Mov}}(X) = \text{Nef}(X)$, we can take any pseudo-effective divisors and any admissible flags of X . Thus, as in Example 3.18, we can describe the limiting Okounkov bodies of any pseudo-effective divisors on X . Let us see a specific example. Let $f = x_0f_0 + x_1f_1$ be the defining equation of X . Also, for a ruled surface $\pi : E \rightarrow C'_0$ with a section C_0 , we assume that $(C_0^2) \geq 0$. Then, by the proof of Lemma ??, $\overline{\text{Eff}}(E) = \text{Nef}(E)$. Fix an admissible flag $X_{\bullet} : X \supset E \supset C \supset \{p\}$, where C is any curve of bidegree (s_1, s_2) in E (i.e., $C = s_1C_0 + s_2F$ with $(C_0^2) \geq 0$, $(F^2) = 0$ and $(C_0 \cdot F) = 1$) and p is any point in C . Let $\textcircled{1} = \langle E, H_2 \rangle$, $\textcircled{2} = \langle H_1, H_2 \rangle$. First, let $D \in \textcircled{1}$ be a pseudo-effective divisor on X . We may let $D = aE + bH_2$ with $a, b \geq 0$ with at least one nonzero a, b . It is clear that $\text{ord}_E(\|D\|) = a$ and $\mu = a + \frac{b}{e}$. Let $A_{x_1} := ((x_1 - a)H_1 + (ae + b - x_1e)H_2)|_E$ (we can describe $A_{x_1} = \alpha C_0 + \beta F$ by using (C_0^2) , $(H_i \cdot C_0)$ and $(H_i \cdot F)$). Note that $\mu_{x_1} = \sup\{\alpha > 0 \mid A_{x_1} - \alpha C \in \text{Big}(E)\}$. Also, $(P_{(P_{D_{x_1}}|_E - x_2C) \cdot C}) = (A_{x_1} \cdot C) - x_2(s_1^2(C_0^2) + 2s_1s_2)$. Therefore, we obtain

$$\Delta_{X_{\bullet}}^{\lim}(aE + bH_2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a \leq x_1 \leq a + \frac{b}{e}, 0 \leq x_2 \leq \mu_{x_1}, \\ 0 \leq x_3 \leq (A_{x_1} \cdot C) - x_2(s_1^2(C_0^2) + 2s_1s_2)\}.$$

Next, let $D \in \textcircled{2}$ be a pseudo-effective divisor on X . We may let $D = aH_1 + bH_2$ with $a, b \geq 0$ with at least one nonzero a, b . In this case, since D is movable, $\text{ord}_E(\|D\|) = 0$. Also, since $D - x_1E = (x_1 + a)H_1 + (b -$

$x_1 e)H_2$, $\mu = \frac{b}{e}$. Let $A_{x_1} := ((x_1 + a)H_1 + (b - x_1 e)H_2)|_E$. As above, note that $\mu_{x_1} = \sup\{\alpha > 0 \mid A_{x_1} - \alpha C \in \text{Big}(E)\}$. Also, $(P_{(P_{D_{x_1}}|_E - x_2 C)} \cdot C) = (A_{x_1} \cdot C) - x_2(s_1^2(C_0^2) + 2s_1 s_2)$. Thus, we obtain

$$\Delta_{X_\bullet}^{\lim}(aH_1 + bH_2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq \frac{b}{e}, 0 \leq x_2 \leq \mu_{x_1}, \\ 0 \leq x_3 \leq (A_{x_1} \cdot C) - x_2(s_1^2(C_0^2) + 2s_1 s_2)\}.$$

(Case 3 : $d = 2$ with $e \geq 2$) Finally, in this case,

$$\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}(eH_2 - H_1) \text{ (FIGURE 4),}$$

$$\text{Nef}(X) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}H_2,$$

where H_1, H_2 are as above.

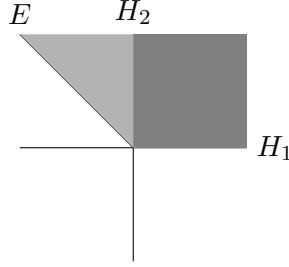


FIGURE 4. $\overline{\text{Eff}}(X)$ in $N^1(X)_{\mathbb{R}}$

Moreover, X has two SQMs; one is the identity, and the other one with its undefined locus is discussed in the proof of [15, Theorem 1.1]. More precisely, let $f = x_0^d f_0 + x_0^{d-1} x_1 f_1 + \cdots + x_1^d f_d$ be the defining bihomogeneous polynomial of X on $\mathbb{P}^1 \times \mathbb{P}^3$. Then, the undefined locus of such SQM is $Z(f_0, \dots, f_d)$. Thus, for any admissible flag $X_\bullet : X \supset S \supset C \supset \{p\}$ with $S \cap Z(f_0, \dots, f_d) = \emptyset$, we can describe the limiting Okounkov bodies of any pseudo-effective divisors on X by Corollary 3.16. We leave the details to the reader.

4. APPLICATION

In this section, we use Corollary 3.16 to obtain conditions of rational polyhedrality of the limiting Okounkov bodies of pseudo-effective divisors on Mori dream threefolds. Before going on, we see some basic facts about Zariski chambers (see [2] for details).

Definition 4.1. Let S be a smooth projective surface. Let P be a nef and big divisor on S . Define

$$\Sigma_P = \{D \in \text{Big}(S) \mid \text{Neg}(D) = \text{Null}(P)\},$$

where $\text{Neg}(D) = \{C \mid C \text{ is an irreducible component of } N_D\}$, where N_D is the negative part of D , and $\text{Null}(P)$ is the set of irreducible curves on S such that $(P \cdot C) = 0$. We call Σ_P the Zariski chambers of P .

Remark 4.2. Note that Σ_P is a convex cone that is neither open nor closed in general. Moreover, for nef and big divisors P and Q , $\Sigma_P \cap \Sigma_Q = \emptyset$ if and only if $\Sigma_P \neq \Sigma_Q$.

The importance of Zariski chambers come from the following Proposition 4.3 which is the main theorem in [2].

Proposition 4.3. *Let S be a smooth projective surface. Then, Zariski chambers give a locally finite decomposition of $\text{Big}(S)$ such that the following holds:*

- (1) *The support of the negative part of the divisors in each chamber is constant.*
- (2) *On each of the chambers, the volume function is given by a single quadratic polynomial.*
- (3) *In the interior of each of the chambers, the stable base loci are constant.*

Remark 4.4. On each Zariski chamber, the map taking a divisor to its negative part is linear.

Now, by using Proposition 4.3 and Corollary 3.16, we obtain the following corollary.

Corollary 4.5. *Notation is as in Corollary 3.16. Let D be any pseudo-effective divisor. Suppose the following conditions hold:*

- (1) *$\text{Big}(S)$ has finitely many Zariski chambers.*
- (2) *$\mu_t := \sup\{\alpha > 0 \mid P_{D_t}|_S - \alpha C \in \overline{\text{Eff}}(S)\}$ is piecewise linear with finitely many pieces.*

Then, $\Delta_{X_\bullet}^{\lim}(D)$ is rational polyhedral.

In particular, if S is of Picard number 1, then $\Delta_{X_\bullet}^{\lim}(D)$ is rational polyhedral.

Proof. Since there are finitely many Mori chambers, we need to show the rational polyhedrality of $\Delta_{X_\bullet}^{\lim}(D)$ for each of them. Fix $i = 1, \dots, r$. First, suppose that (1) and (2) hold. By Remark 3.12-(2), $\Delta_{X_\bullet}^{\lim}(D)_{x_1=t} = \Delta_{S_\bullet}^{\lim}(P_{D_t}|_S) + l_i(t)$ for each $t = t_i \in [\alpha_i, \beta_i]$. Since $\text{Big}(S)$ has finitely many Zariski chambers, by Remark 4.4 and Example 2.14, each $\Delta_{X_\bullet}^{\lim}(A)_{x_1=t_i}$ is rational polyhedral. However, since each μ_t is piecewise linear with finite pieces and $l_i(t)$ is linear, we are done.

Now, let S be of Picard number 1. Then the condition (1) is clear. Also, since P_{D_t} is linear on each Mori chamber and S is of Picard number 1, condition (2) holds. Therefore, $\Delta_{X_\bullet}^{\lim}(D)$ is rational polyhedral. \square

Example 4.6. Again, let X be two points blowing-up of \mathbb{P}^3 . Let $X_\bullet : X \supset S = E_1 \cong \mathbb{P}^2 \supset C \supset \{x\}$ be an admissible flag such that C is a curve on S and x is a point in C . Then, by Corollary 4.5, since S is of Picard number 1, $\Delta_{X_\bullet}^{\lim}(D)$ is rational polyhedral for any pseudo-effective divisor $D \in \textcircled{1}$ or $\textcircled{2}$ on X , where $\textcircled{1}$ and $\textcircled{2}$ are as in Example 3.18.

REFERENCES

- [1] K. Altmann, J. Wiśniewski, Polyhedral divisors of Cox rings, Michigan Math. J. 60 (2011), no. 2, 463-480.
- [2] T. Bauer, A. Küronya, and T. Szemberg, Zariski chambers, volumes, and stable base loci, J. Reine Angew. Math. 576 (2004), 209-233.
- [3] S.R. Choi, Y. Hyun, J. Park, and J. Won, Asymptotic base loci via Okounkov bodies, preprint available at <http://arxiv.org/pdf/1507.00817>

- [4] S.R. Choi, J. Park, and J. Won, Okounkov bodies associated to pseudoeffective divisors, preprint available at <http://arxiv.org/pdf/1508.03922v3.pdf>
- [5] Y. Hu, and S. Keel, Mori dream spaces and GIT, *Michigan Math. J.* 48 (2000), 331-348.
- [6] S. Keicher, Algorithms for Mori dream spaces, PhD Thesis, Universitat Tübingen, 2014, <http://nbn-resolving.de/urn:nbn:de:bsz:21-dspace-540614>.
- [7] A. Küronya, V. Lozovanu, and C. Maclean, Convex bodies appearing as Okounkov bodies of divisors, *Adv. Math.* 229 (2012), no. 5, 2622-2639.
- [8] A. Küronya and V. Lozovanu, Infinitesimal Newton-Okounkov bodies and jet separation, preprint available at <http://arxiv.org/pdf/1507.04339v1.pdf>
- [9] A. Küronya and V. Lozovanu, Positivity of line bundles and Newton-Okounkov bodies, preprint available at <http://arxiv.org/pdf/1506.06525v1.pdf>
- [10] R. Lazarsfeld, *Positivity in Algebraic Geometry I, II*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 48. Springer-Verlag, Berlin, 2004.
- [11] R. Lazarsfeld and M. Mustață, Convex bodies associated to linear series, *Ann. Sci. Éc. Norm. Supér.* (4) 42 (2009), no. 5, 783-835.
- [12] S. Okawa, On images of Mori dream spaces, to appear in *Math. Ann.*
- [13] A. Okounkov, Brunn-Minkowski inequality for multiplicities. *Invent. Math.* 125 (1996), no. 3, 405-411.
- [14] A. Okounkov, Why would multiplicities be log-concave? in *The orbit method in geometry and physics*, *Progr. Math.*, 213, Birkhäuser Boston, Boston, MA, 2003.
- [15] J.C. Ottem, Birational geometry of hypersurfaces in products of projective spaces. *Math. Z.* 280 (2015), no. 1-2, 135-148. 14E05

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJON 305-701, KOREA

E-mail address: `jsshin1991@kaist.ac.kr`